Probability of anomalously large bit-error rate in long haul optical transmission

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We consider a linear model of optical transmission through a fiber with birefringent disorder in the presence of amplifier noise. Both disorder and noise are assumed to be weak, i.e., the average bit-error rate (BER) is small. The probability distribution function (PDF) of rare violent events leading to the values of BER much larger than its typical value is estimated. We show that the PDF has a long algebraiclike tail.

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Optical fibers are widely used for transmission of information. In an ideal case, information carried by pulses would be transmitted nondamaged. In reality, however, various impairments lead to information loss. Noise generated by optical amplifiers and fiber birefringence are the two major impairments in high-speed fiber communications. The amplifier noise is short-correlated in time, while the birefringence varies significantly along the optical line and is practically frozen in time, since the characteristic temporal scale of such variations is long compared to the signal propagation time through the entire fiber line. Coexistence of two different sources of randomness characterized by two well-separated time scales is common in statistical physics of disordered systems. A classical example is the glassy behavior driven by short-correlated thermal noise in a system with frozen structural disorder. (See, e.g., Ref. [1] and later discussion.) Complete statistical description of a system, with both disorder and noise present, requires the two step averaging, and formally introducing a so-called higher-order statistics, i.e., a probability distribution function (second-step averaging over the disorder) of a quantity defined in terms of another probability distribution function (first-step averaging over the noise). Extreme non-Gaussianity of the higher-order statistics is an important feature of the disordered systems [1]. In this paper we show that, first, the disordered system approach is appropriate for optical fiber systems, and second, we report emergence of an extremely non-Gaussian tail in the optical fiber system higher-order statistics.

Birefringent disorder is caused by weak random ellipticity of the fiber cross section. Birefringence splits the pulse into two polarization components and also leads to pulse broadening [2–4]. This effect known as polarization mode dispersion (PMD) have been extensively studied experimentally [5–10] and theoretically [11]. PMD is usually characterized by the so-called PMD vector that was found to obey Gaussian statistics [11]. It was also shown, e.g., in Ref. [12], that first-order PMD compensation corresponding to cancellation of the PMD vector on the carrier frequency is experimentally implementable. Higher-order generalizations of the PMD vector (introduced to resolve a complex frequency dependence of the PMD phenomenon with higher accuracy) as well as a suggestion on how to compensate for PMD in higher orders have been also discussed [14] and implemented experimentally, e.g., in Ref. [13]. Common wisdom hiding behind the standard approach says that one should start with evaluating effects of PMD and amplifier noise separately and then estimate the joint effect taking the impairments on equal footing. In this paper we challenge this equal-footing approach. We show that the overall effects of temporal noise and structural disorder may not be separated since bit-error rate (BER) strongly depends on a realization of birefringent disorder. Thus, the PDF of BER and especially its tail corresponding to large values of BER are the objects of prime interest and practical importance for describing the probability of the system outage.

Our paper is organized as follows. We start with introducing and discussing the equation for an optical pulse evolution in a birefringent fiber also influenced by the amplifier noise. BER produced by the amplifier noise for a given realization of the birefringent disorder is analyzed first. Then we describe the PDF of BER, found by averaging over many realizations of disorder. Some general remarks conclude the paper.

The envelope of the electromagnetic (optical) field propagating through optical fiber in the linear regime (i.e., at relatively low pulse intensity), which is subject to PMD distortion and amplifier noise, satisfies the following equation [2,3,15]:

$$\partial_z \boldsymbol{\Psi} - i\hat{\Delta}(z)\boldsymbol{\Psi} - \hat{m}(z)\partial_t \boldsymbol{\Psi} - i d(z)\partial_t^2 \boldsymbol{\Psi} = \boldsymbol{\xi}(z,t), \qquad (1)$$

z, t, $\boldsymbol{\xi}$, and d being the position along the fiber, the retarded time (measured in the reference frame moving with the optical signal), the amplifier noise, and the chromatic dispersion coefficient, respectively. The envelope $\boldsymbol{\Psi}$ is a two-component complex field where the components stand for two polarization states of the optical signal. Birefringent disorder is characterized by two random Hermitian 2×2 traceless matrices $\hat{\Delta}$ and $\hat{m}$ measuring fiber birefringence in the first and second order and corresponding to the first two terms of the expansion in $\omega-\omega_0$ with ω and ω_0 being the signal and carrier frequencies, respectively. The disorder is frozen at least on all the propagation-related time scales, i.e., the two matrices can be considered to be t independent. The random matrix $\hat{\Delta}(z)$ can be excluded from the consideration using a transformation to the coordinate system rotating together with the signal polarization state at the carrier frequency: $\boldsymbol{\Psi}\to\hat{V}\boldsymbol{\Psi}$, $\boldsymbol{\xi}\to\hat{V}\boldsymbol{\xi}$ and $\hat{m}\to\hat{V}\hat{m}\hat{V}^{-1}$. The unitary matrix $\hat{V}(z)$ is the ordered exponential, $T\exp[i\int_0^z dz'\hat{\Delta}(z')]$ defined as the solution of the equation, $\partial_z\hat{V}=i\hat{\Delta}\hat{V}$, with $\hat{V}(0)=\hat{1}$. Hereafter, we use the notation $\hat{m}$, $\boldsymbol{\xi}$, and $\boldsymbol{\Psi}$ for the renormalized objects whereas the original objects will be no more referred to. With this notation the equation for $\boldsymbol{\Psi}$ has the same form as Eq. (1) if we set $\hat{\Delta}=0$. Note that if no additive noise is taken into account, the resulting renormalized equation description is completely equivalent to the traditional fiber optics linear mode coupling theory approach [11] under the assumption of weak coupling between normal linear modes. The solution of the renormalized equation can be partitioned into a sum of a homogeneous contribution $\boldsymbol{\varphi}$ insensitive to the additive noise and an inhomogeneous component $\boldsymbol{\phi}$:

$$\boldsymbol{\varphi}=\hat{W}(z)\boldsymbol{\Psi}_0(t),\quad \boldsymbol{\phi}=\int_0^z dz'\,\hat{W}(z)\hat{W}^{-1}(z')\boldsymbol{\xi}(z',t),\quad (2)$$

$$\hat{W}(z)=\exp\left[i\int_0^z dz'\,d(z')\partial_t^2\right]T\exp\left[\int_0^z dz'\,\hat{m}(z')\partial_t\right],\quad (3)$$

where $\boldsymbol{\Psi}_0$ describes the input pulse shape (at $z=0$).

We assume the optical system length Z to be much larger than the distance between the nearest amplifier stations (the stations are set to compensate for losses in the pulse intensity). Coarse graining on the interamplifier scale allows treating amplification in the continuous limit. Zero in average additive noise $\boldsymbol{\xi}$ is the amplification leftover. The amplifier noise has Gaussian statistics [16] and its correlation time (set by quantum excitation processes in amplifiers) is much shorter then the pulse width. Therefore, the statistics of ξ is fully determined by its pair correlation function

$$\langle\xi_\alpha(z_1,t_1)\xi_\beta^*(z_2,t_2)\rangle=D_\xi\delta_{\alpha\beta}\delta(z_1-z_2)\delta(t_1-t_2),\quad (4)$$

and the coefficient D_ξ characterizes the noise strength.

Averaging over birefringent disorder is of different nature. Statistics here is collected over different fibers or, alternatively, over different states of the same fiber collected over time (birefringence is known to vary on a time scale substantially exceeding the pulse propagation time). The matrix $\hat{m}$ can be expanded in the Pauli matrices $\hat{m}(z)=h_j(z)\hat{\sigma}_j$ where h_j is a real three-component field. The field is zero in average and short-correlated in z (since the typical scale of the birefringence variations is small compared to the propagation distance Z). It enters the observables described by Eqs. (2) and (3) in an integral form and, according to the central limit theorem, can be treated as a Gaussian random field described by the following pair correlation function:

$$\langle h_i(z_1)h_j(z_2)\rangle=D_m\delta_{ij}\delta(z_1-z_2),\quad (5)$$

where D_m characterizes the disorder strength. Note that even though the original $\hat{m}$ entering Eq. (1) is anisotropic the isotropy of h_j [implied in Eq. (5)] is restored as a result of the $\hat{m}\to\hat{V}\hat{m}\hat{V}^{-1}$ transformation.

We consider the return-to-zero modulation format when pulses in a given frequency channel are well separated in t. Detection of a pulse at the fiber output corresponding to $z=Z$ requires a measurement of the pulse intensity I,

$$I=\int dt G(t)|\mathcal{K}\boldsymbol{\Psi}(Z,t)|^2,\quad (6)$$

where the function $G(t)$ is a convolution of the electrical (current) filter function with the sampling window function (limiting the information slot). The linear operator $\mathcal{K}$ in Eq. (6) stands for an optical filter and may also incorporate a variety of engineering "tricks" applied to the output signal $\boldsymbol{\Psi}(Z,t)$. Ideally, I takes a distinct value if the bit encodes "1" and is negligible if the bit encodes "0." Both the noise and disorder enforce I to deviate from its ideal value. One declares the output signal to encode 0 or 1 if the value of I is less or larger than the decision threshold I_0. The information is lost if the output value of the bit differs from the input one. The probability of such event should be small (this is a mandatory condition for a successful fiber line performance), i.e., both impairments typically cause only small distortion of a pulse. Formally, this means $D_\xi Z\ll 1$, $D_m Z\ll 1$, where the initial signal width and its amplitude are both rescaled to 1.

Below we focus our analysis on the initially "1" bit. We do not consider the evolution of a zero bit since it does not contribute to the anomalously large values of BER, which we are mainly interested to describe. The probability to lose the "1" bit is $B=\int_0^{I_0}dI\,P(I)$. Here $P(I)$ is the PDF of the signal intensity (6) (which fluctuates due to the noise $\boldsymbol{\xi}$) for the initial signal $\boldsymbol{\Psi}_0$ corresponding to the bit "1." In engineering practice BER is measured collecting the statistics over many initially identical pulses (since different pulses sense different realizations of the noise, this averaging over different pulses is actually equivalent to the noise averaging). Repeating the measurement of B many times (each separated from the previous one by a time interval larger than the characteristic time of the disorder variations or, alternatively, making measurements on different fibers) one constructs the PDF $\mathcal{S}(B)$ of B. The PDF achieves its maximum at B_0, a typical value of B. Even though the average distortion of a pulse caused by the noise and disorder is weak, rare but violent events may substantially affect the optical system performance. The probability of such rare events is determined by $\mathcal{S}(B)$ taken at $B\gg B_0$. Our further analysis is focused on the PDF tail.

Mentioning the importance of accounting for an appropriate form of the linear operator $\mathcal{K}$, we restrict our discussion to two possibilities, when the optical filtering is accompanied by an overall time shift (this operation is usually called "setting the clock") or by a compensation achieved by insertion of an additional piece of fiber with adjustable birefringence. An optical filter is required to separate different frequency channels. In addition, the filter smoothes out an otherwise strong impact on the pulse caused by amplifier noise temporal ultra-locality. ''Setting the clock'' procedure is formalized as $\mathcal{K}_{cl}\boldsymbol{\Psi}=\boldsymbol{\Psi}(t-t_{cl})$, where t_{cl} is an optimal time delay. The so-called first-order compensation means $\mathcal{K}_1\boldsymbol{\Psi}=\exp[-H_j\hat{\sigma}_j\partial_t]\boldsymbol{\Psi}$, where $H_j\equiv\int_0^Z dz\, h_j(z)$ and the usual exponent (instead of the ordered one) enters Eq. (3). In the engineering practice or laboratory experiments all three strategies (along with some additional ones) can be applied simultaneously. Below we imply that the optical filter is always inserted. As far as the ''setting the clock'' and the first-order compensation procedures are concerned we intend to compare those effects with the pure (no compensation) case.

The first step in our calculations is finding the value of B for a given disorder realization. In this case the $\boldsymbol{\varphi}$ part of the envelope $\boldsymbol{\Psi}$ is a constant (dependent on the disorder) while the $\boldsymbol{\phi}$ contribution fluctuates. One obtains from Eqs. (2) and (4) that $\boldsymbol{\phi}$ is a zero mean Gaussian variable with the pair correlation function

$$\langle \phi_\alpha(z,t_1)\phi^*_\beta(z,t_2)\rangle = D_\xi z\,\delta_{\alpha\beta}\delta(t_1-t_2).$$

Note that the statistics of $\boldsymbol{\phi}$ is sensitive neither to the chromatic dispersion coefficient d nor to the birefringence matrix $\hat{m}$. Thus, averaging over the noise statistics is reduced to a Gaussian path integral over $\boldsymbol{\phi}$. The inequality $D_\xi Z\ll 1$ justifies the saddle-point evaluation of the integral and also allows estimating $\ln B$ as $\ln P(I_0)$. Therefore, one finds that the product $D_\xi Z\ln B$ is a negative quantity of order 1 insensitive to the noise characteristics. This quantity depends on the initial signal profile Ψ_0, disorder h_j, integral chromatic dispersion $\int_0^Z d(z)dz$, and also on the details of the measurement and compensation procedures.

We next analyze the dependence of B on the birefringent disorder. The smallness of $D_\xi Z$ means that even weak disorder can produce large deviations of B from its typical value B_0 $[\ln B_0\sim-(D_\xi Z)^{-1}]$ corresponding to $h_j=0$. Therefore $\ln(B/B_0)$ can be expanded as a series in h_j. The leading contribution is found upon expanding the ordered exponential (3) [entering $\boldsymbol{\varphi}$ in accordance with Eq. (2)] in a series in $\boldsymbol{h}$, followed by substituting the result for $\boldsymbol{\varphi}$ into Eq. (6), and performing the saddle-point calculations aiming to find $P(I_0)$. The smallness of $D_\xi Z$ enables us to substitute $\ln B$ by $\ln P(I_0)$. Then, in the second order in h, one obtains

$$\ln(B/B_0)=\Gamma/(D_\xi Z),$$

$$\Gamma=\mu_1 H_3+\mu_2 H_j^2+\mu_2'\int_0^Z dz_1\int_0^{z_1}dz_2[h_1(z_1)h_2(z_2) - h_2(z_1)h_1(z_2)], \tag{7}$$

where the initial pulse $\boldsymbol{\Psi}_0$ is assumed to be linearly polarized. The coefficients μ entering Eq. (7) are sensitive to the initial signal profile Ψ_0, chromatic dispersion, and the measurement procedure. These coefficients can be calculated only numerically, however, some conclusions based on the general form of Eq. (7) can be drown without these detailed calculations.

If no compensation is applied to the output signal $\mu_2\sim 1$, whereas μ_1 and μ_2' constitute corrections related to the temporal asymmetry of the pulse (generated by the optical filter) and to the so-called pulse chirp, respectively. If no compensation is applied the leading role is played by the first term in Eq. (7), and one arrives at $\ln\mathcal{P}\approx-\Gamma^2/(2\mu_1^2 D_m Z)$. If the ''setting the clock'' procedure is applied (with t_{cl} dependent on $\boldsymbol{h}$: $tw_{cl}=t_0-H_3$, with t_0 being the optimal time shift without disorder) the first-order term in the right-hand side of Eq. (7) cancels out and the second-order term is dominated by $\mu_2(H_1^2+H_2^2)$. According to Eqs. (2) and (3) the statistics of B depends on the integral dispersion $\int_0^Z dz\, d(z)$. In practice, some special technical efforts (dispersion compensation) are made to reduce it, so that usually this quantity is small. If the initial pulse is real (no chirp) and the integral dispersion is small, then μ_2' is negligible. In this case Eq. (5) leads to the following PDF of Γ:

$$\mathcal{P}(\Gamma)=\frac{1}{2\mu_2 D_m Z}\exp\left(-\frac{\Gamma}{2\mu_2 D_m Z}\right). \tag{8}$$

If the first-order compensation scheme, described by $\mathcal{K}_1$, is used the first two terms in the right-hand side of Eq. (7) cancel out, $\mu_1=\mu_2=0$, and if also $\mu_2'\neq 0$ one arrives at

$$\mathcal{P}(\Gamma)=(2D_m Z\mu_2')^{-1}\cosh^{-1}\left(\frac{\pi\Gamma}{2\mu_2' D_m Z}\right), \tag{9}$$

replacing Eq. (8). Comparing Eqs. (8) and (9) with the result for the no-compensation case we conclude that the compensation leads to much narrower PDF. If μ_2' is also 0 (i.e., the first-order compensation is applied and the output signal is not chirped) the higher-order terms in the expansion of Γ in h_j should be accounted for. This case will be discussed elsewhere.

Let us now analyze the PDF of B, $\mathcal{S}(B)$. Expressing Γ via B according to Eq. (7) followed by substituting the result into Eq. (8) for the PDF of Γ leads to the following expression for the remote, $B\gg B_0$, tail of the PDF of B in the ''setting the clock'' case:

$$\mathcal{S}(B)dB\sim\frac{B_0^\alpha dB}{B^{1+\alpha}},\quad \alpha=\frac{D_\xi}{2\mu_2 D_m}. \tag{10}$$

The applicability range for Eq. (10) is given by $1\gg\Gamma\gg D_m Z$ that transforms into $1/(D_\xi Z)\gg\ln(B/B_0)\gg D_m/D_\xi$. If no compensation is applied, Eq. (10) transforms into $\ln\mathcal{S}\approx-D\xi^2 Z/(2D_m\mu_1^2)$, while in the first-order compensation case explained by Eq. (9), the asymptotic result for the PDF tail (10) remains valid with μ_2 replaced by μ_2'/π. The dependence of the PDF on BER is illustrated in Fig. 1. One also finds from Eq. (10) that the outage probability, $\mathcal{O}\equiv\int_{B_*}^1 dB\,\mathcal{S}(B)$, where B_* is some fixed value taken to be much larger than B_0, is estimated as $\ln\mathcal{O}\sim(D_\xi/D_m)\ln(B_0/B_*)$.

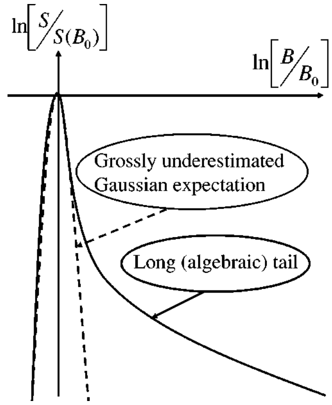

FIG. 1. Schematic log-log plot of the PDF of bit-error rate.

There is also a universal remote tail of $\mathcal{S}(B)$ corresponding to huge fluctuations of the disorder when the signal is almost destroyed by the fluctuations. In this extreme case I is close to I_0 already at $\boldsymbol{\xi}=0$. Then B describes the probability that the inhomogeneous contribution $\boldsymbol{\phi}$ would fill the remaining small gap between I and I_0. Thus one gets $\ln B\sim -\phi^2/(D_\xi Z)$. The value of $\boldsymbol{\phi}$ is proportional to the deviation δh of the h-disorder term from such a special configuration, h_{sp}, which gives $I=I_0$ at $\boldsymbol{\xi}=0$. Thus, $\ln B\sim-(\delta h)^2 Z/D_\xi$. The logarithm of the probability for such disorder configuration to occur is a sum of two contributions: $\sim -1/(D_m Z)$, corresponding to the special configuration, h_{sp}, and the other one $\sim \delta h/D_m$, corresponding to δh. One arrives at the following expression for the probability density:

$$\ln \mathcal{S}(B)\approx -\frac{C_1}{D_m Z}+C_2\sqrt{\frac{D_\xi}{D_m^2 Z}\ln\frac{1}{B}}, \qquad (11)$$

where $C_{1,2}=O(1)$. Equation (11) holds when $(D_m Z)^2 \ll D_\xi Z|\ln B|\ll 1$. Note that the remote tail (11) decays with B faster than the algebraic one (10).

Equation (10) describes our major result. The PDF of BER has a long algebraic tail. The exponent α of the algebraic decay is proportional to the ratio of the amplifier noise variance, D_ξ to the birefringent disorder variance D_m. This statement clearly shows that effects of noise and disorder are, actually, inseparable. Another interesting feature of Eq. (10) is that the exponent α is Z independent. The only Z-dependent factor in the final result (10) is the overall normalization factor B_0^α. Note also that some numerical results, consistent with Eq. (10) are already available. Thus, Fig. 2(a) of Ref. [17] replotted in log-log variables shows a linear relation between $\ln S$ and $\ln B$, i.e., just the algebraic decay predicted by Eq. (10).

Our approach to the fiber optics communication subjected to noise and disorder is close ideologically to the one proposed by Parisi [1] for the Sherrington-Kirpatrick model of spin glasses. Indeed, the PDF $P_J(q)$ for a pair of (many) glassy states to have the overlap q for a given realization of the exchange disorder J corresponds to our $P(I)$ conditioned to the state of the birefringent disorder $\boldsymbol{h}$; $Y_J(q)=\int_q^{q_{max}} dq' P_J(q')$, with q_{max} being the maximal overlap, corresponds to our BER, $B(\boldsymbol{h})$; and, finally, the PDF of $Y_J(q)$, which is the spin glass order parameter, is equivalent to the major object of our study, the PDF of BER. Also note that our analytical statement on the algebraiclike tail of the PDF of BER is equivalent to the algebraic dependence of the PDF of $Y_J(q)$ on $1-Y_J(q)\ll 1$ found for the spin-glass model numerically and also estimated via approximate replica calculations [1]. Finally, we find it useful to stress that our major result that the PDF of BER is characterized by an algebraiclike long tail is universal, i.e., it allows generalizations to a variety of other, e.g., nonlinear, problems in fiber optics where joint effects of noise and disorder are important.

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